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LETTER TO THE EDITOR

Wigner's little group and Berry's phase for massless particles

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Abstract
The 'little group' for massless particles (namely, the Lorentz transformations Λ that leave a null vector invariant) is isomorphic to the Euclidean group E2: translations and rotations in a plane. We show how to obtain explicitly the rotation angle of E2 as a function of Λ and we relate that angle to Berry's topological phase. Some particles admit both signs of helicity, and it is then possible to define a reduced density matrix for their polarization. However, that density matrix is physically meaningless because it has no transformation law under the Lorentz group, even under ordinary rotations.

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Eugene Wigner considered his paper 'On unitary representations of the inhomogeneous Lorentz group' [1] as his most important contribution to physics [2]. The key feature in that article was the introduction of a *little group*, namely a subgroup under which a standard vector s_μ is invariant. For example, a timelike s_μ is (1, 0, 0, 0) and the little group is the familiar rotation group $SO(3)$. A null standard vector can be taken as (1, 0, 0, 1). Spacelike standard vectors have no physical interest.

Massless particles are of prime importance not only in particle physics, but also in quantum information theory: discrete degrees of freedom of photons are the standard physical realisation of quantum bits. To compare data obtained by observers in relative motion, we need the transformation law of these discrete degrees of freedom under Lorentz transformations. This problem has recently been the subject of intensive activity [3, 4]. This letter presents an alternative, more efficient approach to the quantum properties of the little group.

To find explicitly the little group that corresponds to the null s_μ, let us introduce an auxiliary complex null vector m_μ such that [5]

$$m_\mu s^\mu = m_\mu m^\mu = 0 \tag{1}$$

$$m^*_\mu m^\mu = -1 \tag{2}$$

and a real null vector n_μ that satisfies

$$n_\mu m^\mu = 0 \tag{3}$$

$$n_\mu s^\mu = 1. \tag{4}$$

All these properties are manifestly Lorentz invariant. Moreover, they still hold under the transformation

$$m_\mu \to \mathrm{e}^{\mathrm{i}\xi} m_\mu + \gamma s_\mu \tag{5}$$

$$n_\mu \to n_\mu + \mathrm{e}^{\mathrm{i}\xi}\gamma^* m_\mu + \mathrm{e}^{-\mathrm{i}\xi}\gamma m_\mu^* + |\gamma|^2 s_\mu \tag{6}$$

where $\gamma = \alpha + \mathrm{i}\beta$ is any complex number. Therefore the above transformation of m_μ and n_μ is a subgroup of the Lorentz group. We thereby obtain a representation of the generic little group element

$$g = S(\alpha, \beta) R_z(\xi) \tag{7}$$

where the $S(\alpha, \beta)$ form a subgroup which is isomorphic to the translations of a Euclidean plane and $R_z(\xi)$ is a rotation around the origin of that plane, which in this case is also a rotation around the z-axis. We see that the little group is E2, as shown in various ways by other authors [6–9].

In the case of infinitesimal transformations in Minkowski space, we have

$$g = \mathbf{1} + \alpha A + \beta B + \xi M_{12} \tag{8}$$

where $M_{12} \equiv J_3$, and A and B are commuting generators of translations in the 12-plane, for example [10],

$$A = M_{01} + M_{31} \quad \text{and} \quad B = M_{02} + M_{32}. \tag{9}$$

In general, let Λ^μ_ν denote the Lorentz transformation matrix and k^ν an arbitrary momentum. The task is to find the little group matrix W^μ_ν that corresponds to Λ^μ_ν and k^ν, namely

$$W(\Lambda, k) = L^{-1}(\Lambda k)\Lambda L(k) \tag{10}$$

where spacetime indices were omitted for brevity, and $L(k)$ is the standard Lorentz transformation (next equation) that converts s to an arbitrary momentum k. For example, in the case of massive particles for which the little group is $SO(3)$, if Λ is an ordinary rotation then W is the same rotation irrespective of k. However if Λ is a boost, then W does depend on k. Likewise the various terms in (7) may depend on k.

In this letter, we shall examine the group properties of massless particles. For $s = (1, 0, 0, 1)$, the *standard Lorentz transformation*, as defined in [9], is

$$L(k) = R(\hat{\mathbf{k}}) B_z(|\mathbf{k}|) \tag{11}$$

where $B_z(\zeta)$ is a boost along the z-axis with velocity $(1 - \zeta^2)/(1 + \zeta^2)$, and $R(\hat{\mathbf{k}})$ is the standard rotation that carries the z-axis into the direction of the unit vector $\hat{\mathbf{k}}$. Again following [9], if the direction of $\hat{\mathbf{k}}$ is given by spherical angles θ and ϕ, the *standard rotation* $R(\hat{\mathbf{k}})$ consists of a rotation θ around the y-axis, followed by a rotation ϕ around the z-axis:

$$R(\hat{\mathbf{k}}) = R_z(\phi) R_y(\theta). \tag{12}$$

In these formulae, all Lorentz transformations are passive, even if we occasionally use an active wording.

These 'standard' transformations are necessary to define a unique polarization basis for each $\mathbf{k}$. When a general Lorentz transformation Λ brings s to k, the resulting polarization basis is compared to the standard one, and thereby defines the phase rotation ω.

We now prove by a *classical geometric argument* that, if the Lorentz transformation is a pure rotation ($\Lambda = \mathcal{R}$), then it follows from equation (10) that $S(\alpha, \beta)$ is simply the unit matrix, and therefore $\alpha = \beta = 0$. We also give a simple expression for the rotation angle ξ. From the definition of the little group element, we have

$$W(\mathcal{R}, \mathbf{k}) = B_z^{-1}(|\mathbf{k}|)R^{-1}(\mathcal{R}\hat{\mathbf{k}})\mathcal{R}R(\hat{\mathbf{k}})B_z(|\mathbf{k}|). \tag{13}$$

Since the action of $R^{-1}(\mathcal{R}\hat{\mathbf{k}})\mathcal{R}R(\hat{\mathbf{k}})$ leaves the z-axis invariant, it is equivalent to some rotation $R_z(\varpi)$ around that axis,

$$W(\mathcal{R}, \mathbf{k}) = B_z^{-1}(|\mathbf{k}|)R_z(\varpi)B_z(|\mathbf{k}|) = R_z(\varpi) \tag{14}$$

and since $W(\mathcal{R}, k)$ is a special case of (10), it follows that in equation (7) we have $S^\mu_\nu = \delta^\mu_\nu$ and $\alpha = \beta = 0$. Equation (7) also gives $\xi = \varpi$, and the remaining problem is to find the explicit value of this angle.

To clarify the origin of the phase ξ in equation (7), we note that any rotation in a three-dimensional space can be described by two angles that give the direction of the rotation axis, and a third angle that gives the amount of rotation around that axis. A rotation from $\mathbf{k}$ to $\mathbf{q} = \mathcal{R}\mathbf{k}$ can be performed in many ways (denoted below by $R_{\mathbf{qk}}$), in addition to the given $\mathcal{R}$ that we are seeking to decompose. Since all such rotations satisfy

$$\mathcal{R}\mathbf{k} = R_{\mathbf{qk}}\mathbf{k} \qquad \mathcal{R}^{-1}\mathbf{q} = R_{\mathbf{qk}}^{-1}\mathbf{q} \tag{15}$$

the difference between them is a rotation that preserves $\hat{\mathbf{q}}$, if done after $R_{\mathbf{qk}}$, or a rotation that preserves $\hat{\mathbf{k}}$, if done before $R_{\mathbf{qk}}$. In particular, $\mathbf{q} = \mathcal{R}R_{\mathbf{qk}}^{-1}\mathbf{q}$, so that

$$\mathcal{R}R_{\mathbf{qk}}^{-1} = R_{\hat{\mathbf{q}}}(\omega) \tag{16}$$

where $R_{\hat{\mathbf{q}}}(\omega)$ is a rotation around $\hat{\mathbf{q}}$. Among the infinity of possible $R_{\mathbf{qk}}$ we choose

$$R_{\mathbf{qk}} = R(\hat{\mathbf{q}})R^{-1}(\hat{\mathbf{k}}) \tag{17}$$

where $R(\hat{\mathbf{q}})$ and $R(\hat{\mathbf{k}})$ are standard rotations, as in equation (12). It follows that

$$\mathcal{R} = R_{\mathcal{R}\hat{\mathbf{k}}}(\omega)R(\mathcal{R}\hat{\mathbf{k}})R^{-1}(\hat{\mathbf{k}}) \tag{18}$$

where $R_{\mathcal{R}\hat{\mathbf{k}}}(\omega)$ is a rotation around $\mathcal{R}\hat{\mathbf{k}}$, while $R(\mathcal{R}\hat{\mathbf{k}})$ and $R(\hat{\mathbf{k}})$ are the standard rotations that carry the z-axis to $\mathcal{R}\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}$, respectively. We can thus consider equation (16) as the definition of $R_{\mathcal{R}\hat{\mathbf{k}}}(\omega)$.

Substituting this decomposition into equation (14), we obtain

$$W(\mathcal{R}, k) = R^{-1}(\mathcal{R}\hat{\mathbf{k}})R_{\mathcal{R}\hat{\mathbf{k}}}(\omega)R(\mathcal{R}\hat{\mathbf{k}}) = R_z(\varpi) \tag{19}$$

and we conclude that $\xi = \varpi = \omega$.

To obtain the rotation angle under a general Lorentz transformation, we decompose the latter into two rotations and a standard boost B_z along the z-axis [11],

$$\Lambda = \mathcal{R}_2 B_z(\zeta)\mathcal{R}_1 \tag{20}$$

where the parameter ζ is that defined after equation (11). Note that equation (11) was not a general Lorentz transformation: it was the standard Lorentz transformation that is used to define a polarization basis. On the other hand, equation (20) gives a general transformation, which is naturally different.

As shown below, B_z alone does not lead to a phase rotation. Therefore,

$$\xi = \omega_1 + \omega_2 \tag{21}$$

where both ω_1 and ω_2 are due to the rotations and are given by equation (18). On the other hand, α and β are complicated functions of W, which are related to gauge invariance of electromagnetic couplings in a field theory [8, 9].

We now prove that B_z alone induces no phase rotation [3]. Consider a pure boost along the z-axis, $\Lambda = B_z(\zeta)$, and a generic null vector $k = (|\mathbf{k}|, \mathbf{k})$, where

$$\mathbf{k} = |\mathbf{k}|(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{22}$$

We define $q = B_z(\zeta)k = (|\mathbf{q}|, \mathbf{q})$ where

$$\mathbf{q} = |\mathbf{q}|(\sin\theta'\cos\phi, \sin\theta'\sin\phi, \cos\theta'). \tag{23}$$

Note that the angle ϕ is the same for $\mathbf{k}$ and for $\mathbf{q}$. Thus

$$R^{-1}(\hat{\mathbf{q}}) = R_y^{-1}(\theta')R_z^{-1}(\phi). \tag{24}$$

With the help of equations (12) and (24), we now substitute the standard Lorentz transformations $L(k)$ and $L^{-1}(q)$ as defined by (11), into the definition of the little group element, equation (10), to obtain

$$W(B_z(\zeta), k) = B_z^{-1}(|\mathbf{q}|)R_y^{-1}(\theta')B_z(\zeta)R_y(\theta)B_z(|\mathbf{k}|) \tag{25}$$

where we used $R_z^{-1}(\phi)B_z(\zeta)R_z(\phi) = B_z(\zeta)$.

Consider now the effect of the little group element (25) on the spacelike vector $y = (0, 0, 1, 0)$. That vector is not affected by a boost in the z direction, nor by a rotation around the y-axis. Therefore

$$W(B_z(\zeta), k)y = y \tag{26}$$

so that in this case ξ is either 0 or 2π. Since for $\zeta = 0$ we expect $\xi = 0$, by continuity $\xi = 0$ for all ζ.

Note that although $B_z(\zeta)$ alone does not lead to a phase rotation, it can affect the value of ω_2, since it indirectly appears in the definition of $\mathcal{R}_2$. Indeed, if we decompose $\mathcal{R}_2$ as in equation (18), we obtain

$$\mathcal{R}_2 = R_{\mathcal{R}_2\hat{\mathbf{k}}_2}(\omega_2)R(\mathcal{R}_2\hat{\mathbf{k}}_2)R^{-1}(\hat{\mathbf{k}}_2) \tag{27}$$

where $\mathbf{k}_2$ is defined by

$$k_2 = (|\mathbf{k}_2|, \mathbf{k}_2) = B_z(\zeta)\mathcal{R}_1 k. \tag{28}$$

Thus we see that $B_z(\zeta)$ appears in the decomposition of $\mathcal{R}_2$ and therefore affects ω_2.

Up to this point, the discussion and the formalism were purely classical. In quantum theory, one needs the *unitary representations* of the little group, from which those of the complete Lorentz group can be derived. Each irreducible representation corresponds to some species of elementary particles. According to equation (10), the general transformation law is [6–9]

$$U(\Lambda)|\mathbf{k}, \sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}[W(\Lambda, k)]|\mathbf{q}, \sigma'\rangle \tag{29}$$

where $D_{\sigma'\sigma}$ is a unitary representation of the little group and

$$|\mathbf{k}, \sigma\rangle \equiv |\mathbf{k}\rangle \otimes |\sigma\rangle \tag{30}$$

is an appropriate basis. The helicity $\sigma = \mathbf{J}\cdot\hat{\mathbf{k}}$ of a massless particle is Lorentz invariant, so that if we use it for labelling basis states, then the sum in equation (29) consists of a single term, and

$$D_{\sigma'\sigma} = \mathrm{e}^{\mathrm{i}\xi\sigma}\delta_{\sigma'\sigma} \tag{31}$$

where, for a Lorentz transformation which is a pure rotation, ξ is a function of $\mathcal{R}$ and k which is explicitly given by equation (18).

It is experimentally known that some particles, like neutrinos (if they are indeed massless), come with only one sign of helicity. Others, like photons, may have it with both signs, but then the phase angle ξ for them is different. In general, different values of $|\sigma|$ refer to different species of particles, such as photons and gravitons. Within the present formalism, we cannot offer an explanation why, for a given species of particles, half-integral helicities appear with a definite sign. Of course, there can be no helicity-statistics theorem, since we deal with a single particle. These properties must follow from quantum field theory for interacting fields.

An application of the above results is a *direct* derivation of the Berry phase for massless particles. Soon after the introduction of Berry's phase [12] the latter was derived for photons in the adiabatic approximation [13] and then [14] for arbitrary changes in momentum. Finally, derivations that are based on the analysis of connections on Lie groups, and the Poincaré group in particular, were given in [15, 16].

Consider a sequence of rotations that eventually restores the particle momentum to its original value. Its net effect is some *active* rotation around the momentum's direction, $R_{\hat{\mathbf{k}}}(\omega)$. According to equations (19) and (31), the helicity eigenstates acquire phases $-\omega\sigma$, where the minus sign arises from the fact that the transformation law (31) is for passive rotations. To relate this phase to the area on the unit sphere that is enclosed by the orbit of $\hat{\mathbf{k}}$, consider an auxiliary unit vector $\hat{\mathbf{v}}$ in the plane perpendicular to $\hat{\mathbf{k}}$. It is tangent to the sphere at the endpoint of $\hat{\mathbf{k}}$. Let the orthonormal triad $\hat{\mathbf{k}}$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{w}} = \hat{\mathbf{k}} \times \hat{\mathbf{v}}$ be parallel-transported along that orbit. When the latter closes, the orthonormal triad does not return to itself, but ends up as another triad at the same point, which is rotated by the angle ω in the $\hat{\mathbf{v}}\hat{\mathbf{w}}$ plane. Owing to properties of the holonomy group [17], the rotation angle ω is related to the spherical angle Ω by

$$\omega = \int K \, \mathrm{d}S = \int \mathrm{d}\Omega = \Omega \tag{32}$$

where K is the Gaussian curvature, which equals 1 for the unit sphere. The above integral is over the area enclosed by the trajectory of $\hat{\mathbf{k}}$, and the sign of Ω depends on the orientation of the trajectory. We thus obtain

$$\omega = \Omega. \tag{33}$$

An important application of the little group analysis is the meaning of reduced density matrices [18] which are a fundamental concept in quantum information theory. Their properties are significantly modified by relativistic effects [19, 20]. For massless particles that admit both signs of helicity, such as photons, a generic one-particle state is

$$|\Psi\rangle = \int \mathrm{d}\mu\,(k) \sum_\sigma f_\sigma(\mathbf{k})|\mathbf{k}, \sigma\rangle \tag{34}$$

where $\mathrm{d}\mu(k) = \mathrm{d}^3\mathbf{k}/(2\pi)^3(2|\mathbf{k}|)$ is a Lorentz-invariant measure. Then the reduced density matrix for helicity, according to the usual rules, would be

$$\rho_{\sigma\tau} = \int \mathrm{d}\mu\,(k) f_\sigma(\mathbf{k}) f_\tau^*(\mathbf{k}). \tag{35}$$

However, since ξ in equation (31) depends on the photon momentum (even for ordinary rotations) the standard density matrix given by equation (35) has no transformation rule at all. This makes the standard density matrix a useless concept, even when only a fixed reference frame is considered, since any positive operator valued measure (POVM) that describes an experimental set-up must have definite transformation properties at least under ordinary rotations. It is only possible to define an 'effective' density matrix which depends on the detection method [19, 20]. This behaviour contrasts with that of massive particles, for which the little group is $SO(3)$ and reduced density matrices behave properly under rotations, while there is no transformation law only under boosts [21].

The absence of *any* Lorentz transformation law for ρ is due to the fact that the momenta k transform linearly, but the law of transformation of helicity depends explicitly on k. When we compute ρ by summing over momenta, all knowledge of them is lost and it is then impossible to obtain the new ρ by transforming the old one. There is an analogous situation in classical statistical mechanics: a Liouville function can be defined in any Lorentz frame [22], but it has no definite transformation law from one frame to another. Only the complete dynamical system has a transformation law [23].

In summary, we have shown how apparently disparate notions—Wigner's little group and Berry's phase—are closely related. It is curious that the proof made repeated use of ordinary rotations, namely the $SO(3)$ group, which is by itself *another* little group of the Lorentz group.

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